On standard model of Martin's maximum (Interplay between large cardinals and small cardinals)

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1 Introduction

Woodin [5] proved that if the forcing axiom for a poset $\mathbb{Q}$ holds, then for any regular θ with $\mathbb{Q} \in \mathcal{H}_\theta$ there are stationary many $M \in [\mathcal{H}_\theta]^{\omega_1}$ for which an $(M, \mathbb{Q})$-generic filter exists. In this paper we make a remark on this fact in the standard model of Martin's maximum (MM).

First we present the precise statement of the above mentioned fact due to Woodin. For this first we review the notion of $(M, \mathbb{Q})$-generic filter and stationary sets:

Definition 1.1. *Suppose that $\mathbb{Q}$ is a poset.*

(1) Let $\mathcal{D}$ be a set consisting of dense subsets of $\mathbb{Q}$. A filter H on $\mathbb{Q}$ is said to be $\mathcal{D}$-generic if $H \cap D \neq \emptyset$ for any $D \in \mathcal{D}$.

(2) Let M be a set. A subset h of $\mathbb{Q} \cap M$ is called an $(M, \mathbb{Q})$-generic filter if h is a filter on $\mathbb{Q} \restriction (\mathbb{Q} \cap M)$, and $h \cap D \neq \emptyset$ for any dense $D \subseteq \mathbb{Q}$ which belongs to M. (Note that "$h \cap D \neq \emptyset$" is equivalent to "$h \cap D \cap M \neq \emptyset$" because $h \subseteq M$.)

Definition 1.2. *A set X is said to be stationary if for any function $F : [\bigcup X]^{<\omega} \to \bigcup X$ there exists $x \in X$ which is closed under F. For a set A and a regular uncountable cardinal μ, a set $X \subseteq [A]^\mu$ (or $X \subseteq [A]^{<\mu}$) is said to be stationary in $[A]^\mu$ (or in $[A]^{<\mu}$) if X is stationary, and $\bigcup X = A$.*

Remark 1.3. *In this paper we adopt the above notion of stationary sets introduced by Woodin. It slightly differs from the classical definition of stationary subsets of $[A]^{<\mu}$, due to Jech [2]. X is a stationary subset of $[A]^{<\mu}$ in the sense of Jech's classical definition if and only if the set $\{x \in X \mid x \cap \mu \in \mu\}$ is stationary in $[A]^{<\mu}$ in the sense of Def.1.2. Moreover X*

is a stationary subset of $[A]^{<\mu^+}$ *in the sense of Jech's definition if and only if the set* $\{x \in X \mid \mu \subseteq x\}$ *is stationary in* $[A]^{\mu}$ *in the sense of Def.1.2.*

Fact 1.4 (Woodin [5]). *Let* $\mathbb{Q}$ *be a poset, and suppose that the forcing axiom for* $\mathbb{Q}$ *holds, i.e. for every family* $\mathcal{D}$ *of dense subsets of* $\mathbb{Q}$ *with* $|\mathcal{D}| = \omega_1$ *there exists a* $\mathcal{D}$*-generic filter. Then for any* $\mathbb{Q}$ *and any regular uncountable cardinal* θ *with* $\mathbb{Q} \in \mathcal{H}_\theta$ *the set*

$$\{M \in [\mathcal{H}_\theta]^{\omega_1} \mid \omega_1 \subseteq M \ \wedge \ \text{an } (M,\mathbb{Q})\text{-generic filter exists}\}$$

is stationary in $[\mathcal{H}_\theta]^{\omega_1}$.

In this paper we prove the following:

Theorem 1.5. *Suppose that* κ *is a supercompact cardinal in* V. *Let* $\mathbb{P}$ *be the standard revised countable support iteration of length* κ *forcing* MM, *and let* W *be an extension of* V *by* $\mathbb{P}$.

(1) (Veličković) In W, *for any* ω_1*-stationary preserving poset* $\mathbb{Q}$ *and any regular cardinal* θ *with* $\mathbb{Q} \in \mathcal{H}_\theta$ *the set*

$$\{M \in [\mathcal{H}_\theta]^{\omega_1} \mid \omega_1 \subseteq M \ \wedge \ \text{an } (M,\mathbb{Q})\text{-generic filter exists} \wedge M \cap \theta \in V\}$$

is stationary in $[\mathcal{H}_\theta]^{\omega_1}$.

(2) Assume that there are proper class many Woodin cardinals in V. *Then in* W, *for any* ω_1*-stationary preserving poset* $\mathbb{Q}$ *and any regular cardinal* $\theta \geq \kappa^+$ *with* $\mathbb{Q} \in \mathcal{H}_\theta$ *the set*

$$\{M \in [\mathcal{H}_\theta]^{\omega_1} \mid \omega_1 \subseteq M \ \wedge \ \text{an } (M,\mathbb{Q})\text{-generic filter exists} \wedge M \cap \kappa^+ \notin V\}$$

is stationary in $[\mathcal{H}_\theta]^{\omega_1}$.

Thm.1.5 was proved in the course of a joint work with B. Veličković on the following result of Vialle and Weiss [4]:

Theorem 1.6 (Vialle and Weiss [4]). *Assume that* κ *is an inaccessible cardinal and that there exists a poset* $\mathbb{P}$ *with the following properties:*

(i) $\mathbb{P}$ *has the* κ*-covering and the* κ*-approximation properties.*

(ii) $\mathbb{P}$ *forces that* $\kappa = \omega_2$.

(iii) $\mathbb{P}$ *forces the proper forcing axiom (*PFA*).*

Then κ is a strongly compact cardinal. Moreover if there exists a proper $\mathbb{P}$ with the above properties (i)–(iii), then κ is supercompact.

A natural question is whether we need the assumption of properness of $\mathbb{P}$ to obtain supercompactness of κ. Our Thm.1.5 (2) shows some difficulty to drop the assumption of properness. See [4] for details on the relationship between Thm.1.5 (2) and Thm.1.6.

2 Standard iteration for MM

Here we briefly review the standard iteration forcing MM, which is introduced by Foreman-Magidor-Shelah [1].

Let κ be a supercompact cardinal in V.

The iteration is constructed according to a Laver function. Recall that a Laver function is a function $L : \kappa \to \mathcal{H}_\kappa$ such that for any set a and any cardinal $\lambda \geq \kappa$ with $a \in \mathcal{H}_\lambda$ there exists a (κ, λ)-supercompact embedding $j : V \to K$ with $j(L)(\kappa) = a$, where $j : V \to K$ is called a (κ, λ)-supercompact embedding if K is a transitive inner model of ZFC with $K^\lambda \subseteq K$, and j is an elementary embedding whose critical point is κ and such that $j(\kappa) > \lambda$. Recall also that there exists a Laver function $L : \kappa \to \mathcal{H}_\kappa$ if κ is a supercompact cardinal.

Let $L : \kappa \to \mathcal{H}_\kappa$ be a Laver function. Then let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ be the revised countable support iteration such that for each $\beta < \kappa$,

- $\dot{\mathbb{Q}}_\beta = L(\beta)$ if $L(\beta)$ is a $\mathbb{P}_\beta$-name for a semi-proper poset,
- $\dot{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for a trivial forcing notion otherwise.

Then MM holds in $V^{\mathbb{P}_\kappa}$. This follows from the generic elementary embedding argument and the fact below:

Fact 2.1. *In $V^{\mathbb{P}_\kappa}$ every ω_1-stationary preserving poset is semi-proper.*

We call $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ (or $\mathbb{P}_\kappa$) *the standard iteration for* MM *according to L.*

3 Set of models whose traces are in V

In this section we prove Thm.1.5 (1). We use the following lemma:

Lemma 3.1. *Suppose that κ is a supercompact cardinal and that $L : \kappa \to \mathcal{H}_\kappa$ is a Laver function. Let θ be a regular cardinal $> \kappa$, a be an element of $\mathcal{H}_\theta$ and $\mathcal{M}$ be a countable expansion of $\langle \mathcal{H}_\theta, \in, \kappa, a\rangle$. Then there exists $M \in [\mathcal{H}_\theta]^{<\kappa}$ such that*

(i) $M \prec \mathcal{M}$,

(ii) $M \cap \kappa$ is an inaccessible cardinal $< \kappa$,

(iii) if we let $\sigma : M \to \hat{M}$ be the transitive collapse, then

- *$\hat{\theta} := \hat{M} \cap \mathrm{On}$ is a regular cardinal $< \kappa$, and $\hat{M} = \mathcal{H}_{\hat{\theta}}$,*
- *$L(M \cap \kappa) = \sigma(a)$.*

Proof. Let $j : V \to K$ be a $(\kappa, |\mathcal{H}_\theta^V|)$-supercompact embedding such that $j(L)(\kappa) = a$. By the elementarity of j it suffices to show that in K there exists $M \in [\mathcal{H}^K_{j(\theta)}]^{<j(\kappa)}$ such that

(i) $M \prec j(\mathcal{M})$,

(ii) $M \cap j(\kappa)$ is an inaccessible cardinal $< j(\kappa)$,

(iii) if we let $\sigma : M \to \hat{M}$ be the transitive collapse, then

- $\hat{\theta} := \hat{M} \cap \mathrm{On}$ is a regular cardinal $< j(\kappa)$, and $\hat{M} = \mathcal{H}^K_{\hat{\theta}}$,
- $j(L)(M \cap j(\kappa)) = \sigma(j(a))$.

Let $M := j[\mathcal{H}_\theta^V]$. Then $M \in K$, and clearly M satisfies (i) above. Moreover $M \cap j(\kappa) = \kappa$, and hence M satisfies (ii). Finally note that $j \restriction \mathcal{H}_\theta^V$ is the inverse of the transitive collapse of M. Then it can be easily seen that M satisfies (iii) above. □

Now we prove Thm.1.5 (1):

Proof of Thm.1.5 (1). Let κ be a supercompact cardinal, $L : \kappa \to \mathcal{H}_\kappa$ be a Laver function and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ be the standard iteration for MM according to L. Moreover suppose that G_κ is a $\mathbb{P}_\kappa$-generic filter over V, and let $W := V[G_\kappa]$.

In W take an arbitrary ω_1-stationary preserving poset $\mathbb{Q}$, an arbitrary regular cardinal θ with $\mathbb{Q} \in \mathcal{H}_\theta^W$ and an arbitrary function $F : [\mathcal{H}_\theta^W]^{<\omega} \to \mathcal{H}_\theta^W$. All we have to show is that in W there exists $M^* \in [\mathcal{H}_\theta^W]^{\omega_1}$ closed under F such that $\omega_1 \subseteq M^*$, such that an $(M^*, \mathbb{Q})$-generic filter exists and such that $M^* \cap \theta \in V$.

Let $\dot{\mathbb{Q}}$ and $\dot{F}$ be $\mathbb{P}_\kappa$-names of $\mathbb{Q}$ and F, respectively. By Lem.3.1, in V, there exists $M \in [\mathcal{H}_\theta^V]^{<\kappa}$ such that

(i) $M \prec \langle \mathcal{H}_\theta^V, \in, \kappa, L, \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle, \dot{\mathbb{Q}}, \dot{F} \rangle$,

(ii) $\hat{\kappa} := M \cap \kappa$ is an inaccessible cardinal $< \kappa$,

(iii) if we let $\sigma : M \to \hat{M}$ be the transitive collapse, then

- $\hat{\theta} := \hat{M} \cap \mathrm{On}$ is a regular cardinal $< \kappa$, and $\hat{M} = \mathcal{H}_{\hat{\theta}}^V$,
- $L(\hat{\kappa}) = \sigma(\dot{\mathbb{Q}})$.

Here note that $\mathbb{Q}$ is semi-proper in W by Fact 2.1. Then $\mathbb{P}_{\hat{\kappa}} = \sigma(\mathbb{P}_\kappa)$ forces that $L(\hat{\kappa}) = \sigma(\dot{\mathbb{Q}})$ is semi-proper by (i) and (iii) above. Therefore $\dot{\mathbb{Q}}_{\hat{\kappa}} = \sigma(\dot{\mathbb{Q}})$.

We work in W below. Let $G_{\hat{\kappa}}$ be the $\mathbb{P}_{\hat{\kappa}}$-generic filter naturally obtained from G_κ. Then note that the elementary embedding $\sigma^{-1} : \hat{M} \to \mathcal{H}_\theta^V$ can be naturally extended to an elementary embedding $\tau : \hat{M}[G_{\hat{\kappa}}] \to \mathcal{H}_\theta^W$. (For each $\mathbb{P}_{\hat{\kappa}}$-name $\dot{a} \in \hat{M}$ let $\tau(\dot{a}^{G_{\hat{\kappa}}}) := \sigma^{-1}(\dot{a})^{G_\kappa}$.) Note also that

$$M^* := \tau[\hat{M}[G_{\hat{\kappa}}]] = \{\dot{a}^{G_\kappa} \mid \dot{a} \text{ is a } \mathbb{P}_\kappa\text{-name in } M\}.$$

Then $\omega_1 \subseteq M^*$ clearly, and M^* is closed under F because $F \in M^* \prec \langle \mathcal{H}_\theta^W, \in \rangle$. Moreover $M^* \cap \theta = M \cap \theta \in V$. Finally recall that $\dot{\mathbb{Q}}_{\hat{\kappa}} = \sigma(\dot{\mathbb{Q}})$. Hence $\tau(\mathbb{Q}_{\hat{\kappa}}) = \mathbb{Q}$, where $\mathbb{Q}_{\hat{\kappa}} = (\dot{\mathbb{Q}}_{\hat{\kappa}})^{G_{\hat{\kappa}}}$. Let $H_{\hat{\kappa}}$ be the $\mathbb{Q}_{\hat{\kappa}}$-generic filter over $V[G_{\hat{\kappa}}]$ naturally obtained from G_κ. Then $\tau[H_{\hat{\kappa}}]$ is an $(M^*, \mathbb{Q})$-generic filter. Therefore M^* is as desired. □

4 Set of models whose traces are not in V

Here we prove Thm.1.5 (2).

We use the stationary tower forcing, which was introduced by Woodin. First we briefly review basics on the stationary tower forcing. Details can be found in Larson [3].

For a set X and a set $A \supseteq \bigcup X$ let

$$X \uparrow A := \{x \subseteq A \mid x \cap \bigcup X \in X\}.$$

Definition 4.1. *Let μ be an inaccessible cardinal. Then the stationary tower forcing notion $\mathbb{P}_{<\mu}$ is the poset consisting of all stationary $X \in \mathcal{H}_\mu$. For each $X, Y \in \mathbb{P}_{<\mu}$, $X \leq Y$ if the set $X\!\uparrow\!(\bigcup X)\cup(\bigcup Y)\setminus Y\!\uparrow\!(\bigcup X)\cup(\bigcup Y))$ is nonstationary.*

Let μ be an inaccessible cardinal, and suppose that I is a $\mathbb{P}_{<\mu}$-generic filter over V. Then we can construct the ultrapower $\mathrm{Ult}(V, I)$ in $V[I]$:

Let $V^{(<\mu)}$ be the class of all functions $f \in V$ such that $\mathrm{dom}(f) = \mathcal{P}(A)^V$ for some $A \in \mathcal{H}_\mu^V$. For each $f \in V^{(<\mu)}$ and each $A \in \mathcal{H}_\mu^V$ including $\bigcup \mathrm{dom}(f)$ let $f\!\uparrow\! A$ be the function on $\mathcal{P}^V(A)$ such that

$$f\!\uparrow\! A(x) \;=\; f(x \cap \bigcup \mathrm{dom}(f))\,.$$

For each functions $f, g \in V^{(<\mu)}$, letting $A := (\bigcup \mathrm{dom}(f)) \cup (\bigcup \mathrm{dom}(g))$, define

$$f =_I g \;\overset{\text{def}}{\Leftrightarrow}\; \{x \in \mathcal{P}(A)^V \mid f\!\uparrow\! A(x) = g\!\uparrow\! A(x)\} \in I\,.$$

Then $=_I$ is an equivalence relation on $V^{(<\mu)}$. For each $f \in V^{(<\mu)}$ let $(f)_I$ denote the equivalence class represented by f. Moreover for each $(f)_I, (g)_I \in V^{(<\mu)}/=_I$ let

$$(f)_I \in_I (g)_I \;\overset{\text{def}}{\Leftrightarrow}\; \{x \in \mathcal{P}(A)^V \mid f\!\uparrow\! A(x) \in g\!\uparrow\! A(x)\} \in I\,,$$

where $A = (\bigcup \mathrm{dom}(f)) \cup (\bigcup \mathrm{dom}(g))$. It is easy to check that $\in_I$ is well-defined. Let

$$\mathrm{Ult}(V, I) \;:=\; \langle V^{(<\mu)}/=_I\,,\, \in_I\rangle\,.$$

Moreover let

$$\begin{array}{rccc} j_I: & V & \rightarrow & \mathrm{Ult}(V, I) \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & a & \mapsto & (c_a)_I \end{array}$$

where c_a is the function on $\{0\}$ with $c_a(0) = a$ for each $a \in V$. The following is Los' theorem for this ultrapower:

Fact 4.2. *Let μ be an inaccessible cardinal in V, and suppose that I is a $\mathbb{P}_{<\mu}$-generic filter over V. Suppose also that φ is a formula and that $f_1, \ldots, f_n \in V^{(<\mu)}$. Let $A := (\bigcup \mathrm{dom}(f_1)) \cup \cdots \cup (\bigcup \mathrm{dom}(f_n))$. Then*

$$\begin{aligned} &\mathrm{Ult}(V, I) \models \varphi[(f_1)_I, \ldots, (f_n)_I] \\ &\quad\Leftrightarrow\; \{x \in \mathcal{P}(A)^V \mid V \models \varphi[f_1\!\uparrow\! A(x), \ldots, f_n\!\uparrow\! A(x)]\} \in I \end{aligned}$$

Thus $\mathrm{Ult}(V, I)$ is a model of ZFC, and j_I is an elementary embedding.

As for the well-foundedness of ϵ_I, Woodin proved the following:

Fact 4.3 (Woodin). *Assume that μ is a Woodin cardinal in V. Then ϵ_I is well-founded for any $\mathbb{P}_{<\mu}$-generic I filter over V.*

If ϵ_I is well-founded, then we let $\mathrm{Ult}(V, I)$ denote its transitive collapse and j_I denote its composition with the collapsing map. Moreover we let $[f]_I$ denote the image of $(f)_I$ by the collapsing map for each $f \in V^{(<\mu)}$.

Here we give other basic facts on the stationary tower forcing, which are used in the proof of Thm.1.5 (2). The proof can be found in Larson [3]:

Fact 4.4. *Assume that μ is a Woodin cardinal, and suppose that I is a $\mathbb{P}_{<\mu}$-generic filter over V.*

(1) There exists a unique $\nu < \mu$ such that $\nu \in I$. The critical point of j_I is such ν.

(2) $j_I(\mu) = \mu$.

(3) ${}^{<\mu}\mathrm{Ult}(V, I) \cap V[I] \subseteq \mathrm{Ult}(V, I)$.

(4) Suppose that $A \in \mathcal{H}_\mu^V$, and let f be the identity function on $\mathcal{P}(A)^V$. Then $[f]_I = j_I[A]$.

(5) Suppose that A is a transitive set in $\mathcal{H}_\mu^V$. Let $f \in V$ be a function on $\mathcal{P}(A)^V$ such that $f(x)$ is the transitive collapse of x for each $x \subseteq A$. Then $[f]_I = A$.

This finishes a brief review of basics on the stationary tower forcing. Next we give a key lemma to Thm.1.5 (2):

Lemma 4.5. *Let $\mathbb{Q}$ be a poset, θ be a regular cardinal $\geq \omega_3$ with $\mathbb{Q} \in \mathcal{H}_\theta$ and μ be a Woodin cardinal $> \theta$. Assume that the set*

$$\{M \in [\mathcal{H}_\theta]^{\omega_1} \mid \omega_1 \subseteq M \wedge \text{an } (M, \mathbb{Q})\text{-generic filter exists}\}$$

is stationary in $[\mathcal{H}_\theta]^{\omega_1}$. Then for any countable expansion $\mathcal{M}$ of $\langle \mathcal{H}_\theta, \in \rangle$ there exists $X \in \mathbb{P}_{<\mu}$ with the following properties:

(i) $\mathbb{P}_{<\mu} \upharpoonright X$ is ω_1-stationary preserving.

(ii) In $V^{\mathbb{P}_{<\mu} \upharpoonright X}$ there exists $M \in [\mathcal{H}_\theta^V]^{\omega_1}$ such that $\omega_1 \subseteq M \prec \mathcal{M}$, such that an $(M, \mathbb{Q})$-generic filter exists and such that $M \cap (\omega_3)^V \notin V$.

In the proof of this lemma we use the following lemma on the Skolem hull:

Lemma 4.6. *Let θ be a regular uncountable cardinal, Δ be a well-ordering of $\mathcal{H}_\theta$ and $\mathcal{M}$ be a countable expansion of $\langle \mathcal{H}_\theta, \in, \Delta \rangle$. Suppose that $A \in M \prec \mathcal{M}$.*

(1) $\mathrm{Sk}^{\mathcal{M}}(M \cup A) = \{f(a) \mid f : {}^{<\omega}A \to \mathcal{H}_\theta \wedge f \in M \wedge a \in {}^{<\omega}A\}$.

(2) *Suppose that $\mathcal{M}'$ is another countable expansion of $\langle \mathcal{H}_\theta, \in, \Delta \rangle$ and that $M \prec \mathcal{M}'$. Then* $\mathrm{Sk}^{\mathcal{M}}(M \cup A) = \mathrm{Sk}^{\mathcal{M}'}(M \cup A)$.

(3) *For any regular cardinal $\nu \in M$ if $|A| < \nu$, then* $\sup(\mathrm{Sk}^{\mathcal{M}}(M \cup A) \cap \nu) = \sup(M \cap \nu)$.

Proof. (2) and (3) easily follow from (1). We prove (1).

Let N be the set in the right side of the equation. Clearly $\mathrm{Sk}^{\mathcal{M}}(M \cup A) \supseteq N$. We prove that $\mathrm{Sk}^{\mathcal{M}}(M \cup A) \subseteq N$. It is easy to see that $M \cup A \subseteq N$. Thus it suffices to prove that $N \prec \mathcal{M}$.

We use the Tarski-Vaught criterion. Suppose that φ is a formula, that $c^* \in {}^{<\omega}N$ and that $\mathcal{M} \models \exists v \varphi[v, c^*]$. It suffices to find $b^* \in N$ such that $\mathcal{M} \models \varphi[b^*, c^*]$.

Because $c^* \in {}^{<\omega}N$, we can take a function $f : {}^{<\omega}A \to \mathcal{H}_\theta$ in M and $a^* \in {}^{<\omega}A$ such that $c^* = f(a^*)$. Then we define $g : {}^{<\omega}A \to \mathcal{H}_\theta$ as follows: If $a \in {}^{<\omega}A$, and $\mathcal{M} \models \exists v \varphi[v, f(a)]$, then let $g(a)$ be the Δ-least b such that $\mathcal{M} \models \varphi[b, f(a)]$. Otherwise, let $g(a) = 0$.

Then $g \in M$ by the elementarity of M, and so $b^* := g(a^*) \in N$. Moreover $\mathcal{M} \models \varphi[b^*, c^*]$ by the construction of g and the assumption that $\mathcal{M} \models \exists v \varphi[v, c^*]$. Therefore b^* is as desired. □

Proof of Lem.4.5. Suppose that $\mathcal{M}$ is a countable expansion of $\langle \mathcal{H}_\theta, \in \rangle$. We find $X \in \mathbb{P}_{<\mu}$ as in Lem.4.5. We may assume that $\mathcal{M}$ is a countable expansion of $\langle \mathcal{H}_\theta, \in, \Delta, \mathbb{Q} \rangle$, where Δ is some well-ordering of $\mathcal{H}_\theta$.

Let

$$\begin{aligned} Y &:= \{M \in [\mathcal{H}_\theta]^{\omega_1} \mid \omega_1 \subseteq M \wedge \text{an } (M, \mathbb{Q})\text{-generic filter exists}\} \\ X' &:= \{\mathrm{Sk}^{\mathcal{M}}(M \cup \omega_2) \mid M \in Y\} \end{aligned}$$

It is easy to see that X' is stationary in $[\mathcal{H}_\theta]^{\omega_2}$ using Lem.4.6 (2) and the assumption that Y is stationary.

For each $N \in X'$ choose $M_N \in Y$ such that $N = \mathrm{Sk}^{\mathcal{M}}(M_N \cup \omega_2)$. Here note that $M \cap \omega_2 \in \omega_2$ for each $M \in Y$ by the elementarity of M and the fact that $\omega_1 \subseteq M$. Then by Fodor's lemma we can take $\gamma^* < \omega_2$ such that the set

$$X \ := \ \{N \in X' \mid M_N \cap \omega_2 = \gamma^*\}$$

is stationary in $[\mathcal{H}_\theta]^{\omega_2}$. Then $X \in \mathbb{P}_{<\mu}$. We show that this X is as desired.

First we check the property (i) in Lem.4.5. For this take a stationary $S \subseteq \omega_1$ in V and a $\mathbb{P}_{<\mu}$-generic filter I over V containing X. Note that $(\omega_3)^V \in I$. So the critical point of j_I is $(\omega_3)^V$ by Fact 4.4 (1). Then $j_I(S) = S$, and so S is stationary in $\mathrm{Ult}(V, I)$ by the elementarity of j_I. Then S remains to be stationary in $V[I]$ by Fact 4.4 (3).

Next we check the property (ii) in Lem.4.5. Suppose that I is a $\mathbb{P}_{<\mu}$-generic filter over V containing X. In V, for each $N \in X$ let $\sigma_N : N \to \hat{N}$ be the transitive collapse and $\hat{M}_N$ be $\sigma_N[M_N]$. Moreover take a function $f \in V$ on $\mathcal{P}(\mathcal{H}^V_\theta)^V$ such that $f(N) = \hat{M}_N$ for each $N \in X$. Let $M^* := [f]_I$. We prove that M^* witnesses the property (ii) in Lem.4.5.

First we prove that $\omega_1 \subseteq M^* \prec \mathcal{M}$ and that an $(M^*, \mathbb{Q})$-generic filter exists. It suffices to prove that these hold in $\mathrm{Ult}(V, I)$. For each $N \in X$ let $\widehat{\mathcal{M} \restriction N}$ be the transitive collapse of $\mathcal{M} \restriction N$. Take a function $g, h \in V$ on $\mathcal{P}(\mathcal{H}^V_\theta)^V$ such that $g(N) = \widehat{\mathcal{M} \restriction N}$ for each $N \in X$ and such that $h(N) = \sigma_N(\mathbb{Q})$. Then, in V, for every $N \in X$ they hold that $\omega_1 \subseteq f(N) \prec g(N)$ and that an $(f(N), h(N))$-generic filter exists. Thus, by Fact 4.2, in $\mathrm{Ult}(V, I)$, we have that $\omega_1 \subseteq [f]_I \prec [g]_I$ and that an $([f]_I, [h]_I)$-generic filter exists. Here note that $[g]_I = \mathcal{M}$ and that $[h]_I = \mathbb{Q}$ by Fact 4.4 (5). So, in $\mathrm{Ult}(V, I)$, $\omega_1 \subseteq M^* \prec \mathcal{M}$, and an $(M^*, \mathbb{Q})$-generic filter exists.

Next we prove that $M^* \cap (\omega_3)^V \notin V$. First note that

$(*)$ $M^* \cap (\omega_2)^V = j_I(\gamma^*) = \gamma^* < (\omega_2)^V$

because $M_N \cap (\omega_2)^V = \gamma^*$ for each $N \in X$, and $\gamma^* < (\omega_3)^V = \mathrm{crit}(j_I)$. Note also that $\sup(M_N \cap (\omega_3)^V) = N \cap \omega_3$ for each $N \in X$ by Lem.4.6 (3). Therefore

$(**)$ $\sup(M^* \cap (\omega_3)^V) = (\omega_3)^V$

by Fact 4.4.

For the contradiction assume that $M^* \cap (\omega_3)^V \in V$. Then by $(**)$ we can take $\delta \in M^* \cap (\omega_3)^V$ such that $|M^* \cap \delta|^V = (\omega_2)^V$. Let $\tau : \delta \to (\omega_2)^V$ be the Δ-least injection. Then $\tau \in M^*$ because $M^* \prec \mathcal{M}$. So M^* is closed under τ. Then $|M^* \cap (\omega_2)^V|^V = (\omega_2)^V$. This contradicts $(*)$.

Now we have proved that X satisfies the properties (i) and (ii) in Lem.4.5. This completes the proof. $\square$

Now we prove Thm.1.5 (2):

Proof of Thm.1.5 (2). In V let κ be a supercompact cardinal, $L : \kappa \to \mathcal{H}_\kappa$ be a Laver function and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ be the standard iteration for MM according to L. Let G_κ be a $\mathbb{P}_\kappa$-generic filter over V. In $V[G_\kappa]$ suppose that $\mathbb{Q}$ is an ω_1-stationary poset, that θ is a regular cardinal $\geq \kappa^+$ with $\mathbb{Q} \in \mathcal{H}_\theta$ and that $\mathcal{M}$ is a countable expansion of $\langle \mathcal{H}_\theta, \in \rangle$. In $V[G_\kappa]$ we will find $M \in [\mathcal{H}_\theta]^{\omega_1}$ such that $\omega_1 \subseteq M \prec \mathcal{M}$, such that an $(M, \mathbb{Q})$-generic filter exists and such that $M \cap \kappa^+ \notin V$.

In V take a Woodin cardinal $\mu > \theta$. Then μ remains to be a Woodin cardinal in $V[G_\kappa]$. Note that the assumption of Lem.4.5 holds in $V[G_\kappa]$ for $\mathbb{Q}$ and θ by the fact that $V[G_\kappa] \models$ MM and Fact.1.4. In $V[G_\kappa]$ take $X \in \mathbb{P}_{<\mu}$ witnessing Lem.4.5 for $\mathcal{M}$, and let $\mathbb{R} := \mathbb{P}_{<\mu} \restriction X$. Note that $\mathbb{R}$ is semi-proper by Fact 2.1. In V let $\dot{\mathcal{M}}$, $\dot{\mathbb{Q}}$ and $\dot{\mathbb{R}}$ be $\mathbb{P}_\kappa$-names of $\mathcal{M}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively. Moreover, in V, take a sufficiently large regular cardinal $\chi > \mu$.

Then by Lem.3.1 in V we can take $N \in [\mathcal{H}_\chi^V]^{<\kappa}$ with the following properties:

(i) $N \prec \langle \mathcal{H}_\chi^V, \in, \kappa, L, \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle, \theta, \dot{M}, \mu, \dot{\mathbb{Q}}, \dot{\mathbb{R}} \rangle$,

(ii) $\hat{\kappa} := N \cap \kappa \in \kappa$ is an inaccessible cardinal $< \kappa$,

(iii) if we let $\sigma : N \to \hat{N}$ be the transitive collapse, then

- $\hat{\chi} := \hat{N} \cap \mathrm{On}$ is a regular cardinal $< \kappa$, and $\hat{N} = \mathcal{H}_{\hat{\chi}}^V$,
- $L(\hat{\kappa}) = \sigma(\dot{\mathbb{R}})$.

Let $G_{\hat{\kappa}}$ be the $\mathbb{P}_{\hat{\kappa}}$-generic filter over V naturally obtained from G_κ, and let

$$\hat{\theta} := \sigma(\theta)\ ,\ \hat{\mathcal{M}} := \sigma(\dot{\mathcal{M}})_{G_{\hat{\kappa}}}\ ,\ \hat{\mathbb{Q}} := \sigma(\dot{\mathbb{Q}})_{G_{\hat{\kappa}}}\ ,\ \hat{\mathbb{R}} := \sigma(\dot{\mathcal{R}})_{G_{\hat{\kappa}}}\ .$$

Then the elementary embedding $\sigma^{-1} : \hat{N} \to \mathcal{H}_\chi^V$ can be extended to the elementary embedding $\tau : \hat{N}[G_{\hat{\kappa}}] \to \mathcal{H}_\chi^{V[G_\kappa]}$. (Let $\tau(\dot{a}^{G_{\hat{\kappa}}}) := \sigma^{-1}(\dot{a})^{G_\kappa}$ for each $\mathbb{P}_{\hat{\kappa}}$-name $\dot{a} \in \hat{N}$.) Furthermore $\tau(\hat{\mathcal{M}}) = \mathcal{M}$, $\tau(\hat{\mathbb{Q}}) = \mathbb{Q}$, and $\tau(\hat{\mathbb{R}}) = \mathbb{R}$.

Here note that $\dot{\mathbb{Q}}_{\hat{\kappa}} = \sigma(\dot{\mathbb{R}})$ because $L(\hat{\kappa}) = \sigma(\dot{\mathbb{R}})$ is a $\mathbb{P}_{\hat{\kappa}}$-name for a semi-proper poset by the properties of N. Let $H_{\hat{\kappa}}$ be the $\hat{\mathbb{R}}$-generic filter over $V[G_{\hat{\kappa}}]$ naturally obtained from G_κ. Then, by the choice of $\mathbb{R}$ and the elementarity of τ, in $\hat{N}[G_{\hat{\kappa}} * H_{\hat{\kappa}}]$ we can take a set $\hat{M}$ of size ω_1 such that $\omega_1 \subseteq \hat{M} \prec \hat{\mathcal{M}}$, such that an $(\hat{M}, \hat{\mathbb{Q}})$-generic filter exists and such that $\hat{M} \cap (\hat{\kappa}^+)^{\hat{N}} \notin \hat{N}$. Note that $\hat{M} \cap (\hat{\kappa}^+)^{\hat{N}} \notin V$ because $\hat{N} = \mathcal{H}^V_{\hat{\chi}}$.

Let $M := \tau[\hat{M}]$. Then $M \in V[G_\kappa]$. Moreover, in $V[G_\kappa]$, it is easy to see that $\omega_1 \subseteq M \prec \mathcal{M}$, that an $(M, \mathbb{Q})$-generic filter exists and that $M \cap \kappa^+ \notin V$. The last one follows from the facts that $\tau((\hat{\kappa}^+)^{\hat{N}}) = \kappa^+$, that $\hat{M} \cap (\hat{\kappa}^+)^{\hat{N}} \notin V$ and that $\tau \restriction \mathrm{On} = \sigma^{-1} \restriction \mathrm{On} \in V$. Therefore M is as desired. $\square$

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